6- Arithmetic Sequence

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(6) aRithmetic sequence

A sequence is simply a list of numbers. In a sequence each number is called a term of the sequence. There is a ***first*** term, ***second*** term, ***third*** term, and so on. A sequence can be finite in which it is ***possible to count the number*** of terms, or infinite in which the terms ***continue forever.*** → ∞

...has an end.

For example: 1, 3, 6, 10 is a(n) finite sequence

1, 3, 6, 10, ... is a(n) infinte sequence

A sequence is a function whose domain is a set of positive integers. However, a sequence is written using subscript notation rather than function notation.

→ set of x-values

For example: $a_1, a_2, a_3, \ldots, a_n$ ← "n" number of terms.

The subscript identifies the term of the sequence. For instance a_3 is the third term, and a_n is the nth term of the sequence. The entire sequence is usually denoted by $\{a_n\}$.

Sequence

A **finite sequence** is ... the sequence will end at "n"

$\{1, 2, 3 \ldots n\}$, represents some #

An **infinite sequence** is ... the sequence is all natural numbers

$\{1, 2, 3 \ldots \infty\}$

Example 1 Write the first four terms of the sequence. a_1, a_2, a_3, a_4

a) $a_n = \frac{n+1}{n}$

$a_1 = \frac{1+1}{1}$, $a_2 = \frac{2+1}{2}$, $a_3 = \frac{3+1}{3}$, $a_4 = \frac{4+1}{4}$

$\boxed{= 2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}}$

b) $b_n = 2n - 3$

$b_1 = 2(1)-3$, $b_2 = 2(2)-3$, $b_3 = 2(3)-3$, $b_4 = 2(4)-3$

$\boxed{= -1, 1, 3, 5}$

c) $t_n = 2^n$

$t_1 = 2^1$, $t_2 = 2^2$

$= 2, 4,$

$\boxed{*Q \# 1, 2}$

~~Another way of defining a sequence is to define the first term, or the first few terms, and specify the nth term by a formula involving the preceding term(s). Sequences defined in this manner are called ________________~~

~~**Example 2** Write the first four terms of the **recursive formula**: $a = 3$, $a_n = \frac{a_{n-1}}{n}$.~~

Sigma Notation

It is often important to find the **sum of a sequence**, $\{t_n\} = a_1 + a_2 + a_3 + \cdots + a_n$. ← n number of terms.

The expanded notation $(a_1 + a_2 + a_3 + \cdots + a_n)$ can be written more compactly using "Sigma Notation".

The ***Greek letter*** Σ ***(sigma)*** is used as the summation symbol in sigma notation.

$$\underbrace{a_1 + a_2 + a_3 + \cdots + a_n}_{\text{expanded notation}} = \sum_{k=1}^{n} \boxed{a_k}$$

n ← where it ends.
a_k ← formula for #'s in seq.
k=1 ← where the sequence starts.

The integer k is called the ***index of the sum***, which shows where the summation starts.

The integer n shows where the ***summation ends***.

The summation $\sum_{k=1}^{n} a_k$ has $n - k + 1$ terms.

Example 3 Find the sum of each sequence.

a) $\sum_{k=1}^{4}(2k+1)$ n (end), start

$\underset{\text{term 1}}{(2(1)+1) = 3}$, $\underset{\text{term 2}}{(2(2)+1) = 5}$, $\underset{\text{term 3}}{(2(3)+1) = 7}$, $\underset{\text{term 4}}{(2(4)+1) = 9}$

sum of terms 1 → 4, add.

$\Sigma = 3+5+7+9 \boxed{= 24}$

b) $\sum_{k=1}^{5}(k^2+1)$ n (end), start

sum of terms 1 → 5

$(k=1, k=2, k=3, k=4, k=5)$

$\underset{\text{term 1}}{(1)^2+1 = 2}$, $\underset{\text{term 2}}{(2)^2+1 = 5}$, $\underset{t_3}{(3)^2+1 = 10}$, $\underset{t_4}{(4)^2+1 = 17}$, $\underset{t_5}{(5)^2+1 = 26}$

$\therefore \Sigma = 2+5+10+17+26 = \boxed{60}$

c) $\sum_{k=1}^{3}(k^3-k)$ n (end) last term, start

sum of terms 1 → 3

$\underset{\text{term 1}}{(1)^3-1 = 0}$, $\underset{\text{term 2}}{(2)^3-2 = 6}$, $\underset{\text{term 3}}{(3)^3-3 = 24}$

$\therefore \Sigma = 0+6+24 = 30$

Example 4 Write the sum using sigma notation.

a) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{12}{12+1}$

$\frac{1}{1+1} = \frac{1}{2}$ — k, First term k=1

This shows the formula; last term n = 12

$$\boxed{\sum_{k=1}^{12}\left(\frac{k}{k+1}\right)}$$

b) $\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots + \left(\frac{2}{3}\right)^n$

$\left(\frac{2}{3}\right)^1 = \frac{2}{3}$, $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$, $\left(\frac{2}{3}\right)^3$

First term k=1; formula; Last term = n (will go to ∞)

$$\boxed{\sum_{k=1}^{n}\left(\frac{2}{3}\right)^k}$$

38

Arithmetic Sequence

When the difference between successive terms of a sequence is ***always the same number***, the sequence is called arithmetic.

For example the sequence 3, 7, 11, 15, ... is arithmetic because adding 4 to any term produces the next term. The common difference, d, of this sequence is 4. (+4 +4 +4)

~~To develop a formula to find the general term of an arithmetic sequence, the first few terms need to be expanded.~~

~~1st term: $a_1 = a_1$~~
~~2nd term: $a_2 = a_1 + d$~~
~~3rd term: $a_3 =$~~
~~4th term: $a_4 =$~~

Notice that the coefficient of d is one less than the subscript of the term.

> **The nth Term of an Arithmetic Sequence**
>
> For an arithmetic sequence $\{t_n\}$ whose first term is a, with common difference d:
>
> $$t_n = a + (n-1)d$$
>
> for any integer $n \geq 1$
>
> a ← 1st term

Example 5 For each arithmetic sequence, identify the common difference. $= d$

a) 3, 5, 7, 9, ... (+2) $d = +2$

b) 11, 8, 5, 2, ... (−3 −3 −3) $d = -3$

Example 6 Determine if the sequence $\{t_n\} = \{3 - 2n\}$ is arithmetic. → common difference.

① Solve for t_1, t_2, t_3 ...... first 3 terms.

$t_1 = (3 - 2(1)) = 1$
$t_2 = (3 - 2(2)) = -1$) −2
$t_3 = (3 - 2(3)) = -3$) −2

39

② Is there a common difference? yes, $d = -2$ $\therefore$ is arithmetic sequence

n = 12   +3 +3   12th term in

common difference: arithmetic sequence

$n=12$

Example 7 Find the 12th term of the arithmetic sequence 2, 5, 8, ... .35 (12th term in sequence)

$+3 \; +3$

$d = 3$ — 1st term: $a = 2$

$$\boxed{t_n = a + (n-1)d}$$

$$t_{12} = 2 + (12-1)3$$

$$t_{12} = 2 + (11)3$$

$$t_{12} = 2 + 33$$

$$\boxed{t_{12} = 35}$$

Example 8 Which term in the arithmetic sequence 4, 7, 10, ... has a value of 439?

$+3 \; +3$

a = 1st term $\quad d = +3$

$t_n = 439$ — which # term in the sequence is 439

$$\boxed{t_n = a + (n-1)d}$$

$$439 = 4 + (n-1)3$$

$$\frac{435}{3} = \frac{(n-1)\cancel{3}}{\cancel{3}}$$

$$145 = n - 1$$

$$\boxed{146 = n}$$

4, 7, 10 439

1st term — 146th term

BONUS **Example 9** The 7th term of an arithmetic sequence is 78, and the 18th term is 45. Find the first term. → find "a"

$t_n = a + (n-1)d$ ... we also don't know d.

① Know: $\begin{cases} t_7 = a + (7-1)d = 78 \\ t_{18} = a + \boxed{(18-1)d} = 45 \end{cases} \times(-1)$

This just became a system of equations.

②
$$a + 6d = 78$$
$$-a - 17d = -45$$
$$\emptyset \quad -11d = 33$$
$$\boxed{d = -3}$$

③ substitute + solve for a

$$a + 6d = 78$$
$$a + 6(-3) = 78$$
$$a - 18 = 78$$
$$\boxed{a = 96}$$

This means the 1st term is 96

BONUS **Example 10** Find x so that $3x+2$, $2x-3$, and $2-4x$ are consecutive terms of an arithmetic sequence.

Homework ASSIGNMENT # 6

Questions ~~#1, 2-11 (every 2nd letter), 12-14~~

#1, 2-3 (every 2nd letter)
5-9 (every 2nd letter)

Exercise Set

1. Fill in the blanks.
   a) The domain of a sequence is the set of consecutive ____________ numbers.
   b) A sequence with a last term is a(n) ____________ sequence.
   c) A sequence with no last term is a(n) ____________ sequence.
   d) The sequence $a_1 = 2,\ a_n = 2a_{n-1}$ is a ____________ sequence.
   e) The formula for the nth term of an arithmetic sequence is $t_n =$ ____________ .

2. Write the first four terms of each sequence.
   a) $\{n^2 - 2\}$ b) $\left\{\frac{n+2}{n+1}\right\}$
   c) $\{(-1)^{n+1}n^2\}$ d) $\left\{\frac{3^n}{2^n+1}\right\}$
   e) $\left\{\frac{2^n}{n^2}\right\}$ f) $\left\{\left(\frac{2}{3}\right)^n\right\}$

3. Write the nth term of the suggested pattern.
   a) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ b) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$
   c) $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \ldots$ d) $2, -4, 6, -8, \ldots$

4. Write the first four terms of the recursive sequence.
   a) $a = 4,\ t_n = 2 + t_{n-1}$ b) $a = 3,\ t_n = n - t_{n-1}$
   c) $a = 2,\ a_2 = 3,\ a_n = a_{n-1} + a_{n-2}$ d) $a_1 = -1,\ a_2 = 1,\ a_n = na_{n-1} + a_{n-2}$

Assignment #6 KEY

1. a) natural b) finite c) infinite d) recursive e) $t_n = a + (n-1)d$
2. a) $-1, 2, 7, 14$ b) $\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}$ c) $1, -4, 9, -16$ d) $1, \frac{9}{5}, 3, \frac{81}{17}$ e) $2, 1, \frac{8}{9}, 1$ f) $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}$
3. a) $\frac{1}{n}$ b) $\frac{1}{2^{n-1}}$ c) $\left(\frac{2}{3}\right)^n$ d) $(-1)^{n+1}\cdot 2n$
4. a) 4, 6, 8, 10 b) $3, -1, 4, 0$ c) 2, 3, 5, 8 d) $-1, 1, 2, 9$
5. a) 20 b) 22 c) 50 d) 32 e) 15 f) 194
6. a) $\sum_{k=1}^{4}(2k-1)$ b) $\sum_{k=1}^{5}k^2$ c) $\sum_{k=1}^{n}\frac{k}{k+1}$ d) $\sum_{k=1}^{n}\frac{5^k}{k}$
7. a) 7, 11, 15, 19, 23 b) 15, 12, 9, 6, 3 c) 4, 6, 8, 10, 12 d) $-1, -4, -7, -10, -13$
   e) $-5, -\frac{23}{4}, -\frac{13}{2}, -\frac{29}{4}, -8$ f) $-\frac{2}{3}, -\frac{7}{15}, -\frac{4}{15}, -\frac{1}{15}, \frac{2}{15}$
8. a) 38 b) $-\frac{4}{3}$ c) $\frac{15}{4}$ d) -21.25 e) 1.2 f) -25.75
9. a) 13 b) 18 c) 44 d) 18 e) 33 f) 42
10. a) -5 b) -4 c) 47 d) 15 e) 21 f) $40\frac{1}{3}$
11. a) $-\frac{3}{2}$ b) 1 c) -4 d) -2 e) $-3, 4$ f) $-3, \frac{1}{2}$

5. Find the sum of each sequence.

a) $\sum_{k=1}^{5} 4$ **b)** $\sum_{k=1}^{4} (k^2 - 2)$

c) $\sum_{k=2}^{5} (k^2 - 1)$ **d)** $\sum_{k=0}^{3} (k^3 - 1)$

e) $\sum_{k=1}^{4} \frac{k^2}{2}$ **f)** $\sum_{k=6}^{8} (k+1)^2$

6. Express each sum using summation notation with index $k = 1$.

a) $1 + 3 + 5 + 7$ **b)** $1^2 + 2^2 + 3^2 + 4^2 + 5^2$

c) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n}{n+1}$ **d)** $5 + \frac{5^2}{2} + \frac{5^3}{3} + \cdots + \frac{5^n}{n}$

7. Write the first five terms of each arithmetic sequence.

a) $7, 11, 15,$ ____ , ____ **b)** $15, 12, 9,$ ____ , ____

c) $a = 4,\ d = 2$ **d)** $a = -1,\ d = -3$

e) $a = -5,\ d = -\frac{3}{4}$ **f)** $a = -\frac{2}{3},\ d = \frac{1}{5}$

8. Find the indicated arithmetic term.

a) $a = 5,\ d = 3$; find t_{12} **b)** $a = \frac{2}{3},\ d = -\frac{1}{4}$; find t_9

c) $a = -\frac{3}{4},\ d = \frac{1}{2}$; find t_{18} **d)** $a = 2.5,\ d = -1.25$; find t_{21}

e) $a = -0.75,\ d = 0.05$; find t_{40} **f)** $a = -1\frac{3}{4},\ d = -\frac{2}{3}$; find t_{37}

9. Find the number of terms in each arithmetic sequence.

a) $a = 6,\ t_n = -30,\ d = -3$ **b)** $a = -3,\ t_n = 82,\ d = 5$

c) $a = 0.6,\ t_n = 9.2,\ d = 0.2$ **d)** $a = -0.3,\ t_n = -39.4,\ d = -2.3$

e) $-1, 4, 9, \ldots, 159$ **f)** $23, 20, 17, \ldots, -100$

10. Find the first term in the arithmetic sequence.

a) 6th term is 10; 18th term is 46 **b)** 4th term is 2; 18th term is 30

c) 9th term is 23; 17th term is − 1 **d)** 5th term is 3; 25th term is − 57

e) 13th term is − 3; 20th term is − 17 **f)** 11th term is 37; 26th term is 32

11. Find x so that the values given are consecutive terms of an arithmetic sequence.

a) $x + 3,\ 2x + 1$, and $5x + 2$ **b)** $2x,\ 3x + 2$, and $5x + 3$

c) $x - 1,\ \frac{1}{2}x + 4$, and $1 - 2x$ **d)** $2x - 1,\ x + 1$, and $3x + 9$

e) $x + 4,\ x^2 + 5$, and $x + 30$ **f)** $8x + 7,\ 2x + 5$, and $2x^2 + x$

12. If t_n is a term of an arithmetic sequence, what is $t_n - t_{n-1}$ equal to?

13. List the first seven numbers of the Fibonacci sequence $a_1 = 1,\ a_2 = 1,\ a_n = a_{n-1} + a_{n-2},\ n > 2$.

14. The starting salary of an employee is \$23 750. If each year a \$1250 raise is given, in how many years will the employee's salary be \$50 000?

15. An auditorium has 8 seats in the first row. Each subsequent row has 4 more seats than the previous row. What row has 140 seats?

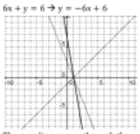
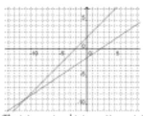
16. A well drilling company charges \$8.00 for the first meter, then \$8.75 for the second meter, and so on in an arithmetic sequence. At this rate, what would be the cost to drill the last meter of a well 120 meters deep?

17. It is said that during the last weeks of his life Abraham deMoivre needed 15 minutes more sleep each night, and when he needed 24 hours sleep he would die. If he needed 8 hours sleep on September 1, what day did he die?

18. The first three terms of an arithmetic sequence are $x - 3$, $\frac{x^2}{25} + 9$, and $3x - 11$. Determine the fourth term.

19. The first, third, and fifth terms of an arithmetic sequence are $2x - 1$, $x^2 - 3$, and $11 - x^2$ respectively. Determine the second term.

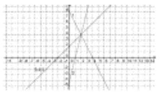
Answers:

1. He should estimate his earnings from sales.
2. On graph pg 5.
3. This is the ordered pair that represents the sales that would produce equivalent earnings.
4. When Jazhon sells more than \$40 000 of concrete.
5. (2,4) is a solution because it **satisfies** both equations in the **system**.
6. yes
7. no
8. yes
9. yes
10. yes
11. no
12. If the coordinates satisfy both (all) equations in the system. Also, the point will be on both lines when graphed.
13. (0,1). Plot each equation using slope and y-intercept. Find the coordinates of the point of intersection.
14. (1,2)
15. (2,4)
16. (2,3)
17. (3,1)
18. (5,3)
19. $(2, -1)$
20. (2,3)
21. (0,10)
22. (5,6)
23. No solution. Parallel lines never intersect.
24. Both lines share all points. We say there are infinite solutions.
25. Both lines share all points. We say there are infinite solutions.
26. Same slope, different y-intercept.
27. Same slope, same y-intercept. Same line.
28. Same slope, same y-intercept. Same line.
29. Answers will vary.
    One solution: lines will have diff. slopes.
    No solutions: Parallel lines.
    Infinite solutions: same lines.
30. One. These equations have different slopes.
31. One solution.
32. One solution.
33. One solution.
34. One solution.
35. No solutions.
36. No solutions.
37. $k = 1$
38. $k = 4$
39. $k = -\frac{5}{12}$
40. $b = 2$
41. $b = -28$
42. $b = \frac{3}{2}$
43. (0,2)
44. Consistent
45. (1,0)
46. $6x + y = 6 \rightarrow y = -6x + 6$

47. The new line passes through the solution to the original system.
48.

49. The intercept and intersection points are not integers therefore difficult to read on the graph. See page 12.
50. $(-11, -9)$
51. The point (or sometimes points) that satisfies all the equations.
52. C
53. $\left(\frac{2}{3}, \frac{1}{3}\right)$
54. $\frac{1}{3} = 2\left(\frac{2}{3}\right) - 1 \rightarrow \frac{1}{3} = \frac{4}{3} - \frac{3}{3} \rightarrow \frac{1}{3} = \frac{1}{3}$

    $\frac{1}{3} = -\left(\frac{2}{3}\right) + 1 \rightarrow \frac{1}{3} = \frac{-2}{3} + \frac{3}{3} \rightarrow \frac{1}{3} = \frac{1}{3}$

    Both equations satisfied by the point $\left(\frac{2}{3}, \frac{1}{3}\right)$.
55. Answered on page.
56. Substitute the point back into the original equations.
57. See #54 above.
58. $\left(-\frac{8}{7}, \frac{31}{7}\right)$
59. (2,7)
60. $(-1, 3)$
61. $(3, -2)$
62. $\left(-6, \frac{5}{2}\right)$
63. $(4, -5)$
64. (1,3)
65. (3,3)
66. (2,1)
67. (20,10)
68. $x + y = 65$
    $x = y + 17$
69. (41,24)
70. $x + y = 102$
    $x = y - 12$
71. (45,57)

72. $a + d = 12$
$5a + 7d = 70$
73. 7 athletic, 5 dress
74. Yes, it satisfies both equations.
75. $4x - 2y = 10$ and $4x - 8y = 4$
76. Yes
77. $4x - 5y = 7$
78. Yes
79. The solution to the original system will be a solution to the new equations too.
80. The solution to the original system will be a solution to the new equations too.
81. Yes. You must multiply each term in an equation by the same constant, but different equations can be multiplied by different constants without affecting the solution.
82. See graph below (with Q89).
83. See graph below.
84. ③ $2y = 2x + 4$
or $y = x + 2$
85. See graph below.
86. ④ $3y = 3x + 6$
or $y = x + 2$
87. See graph below.
88. ⑤ $5y = 5x + 10$
or $y = x + 2$
89. See graph below.



90. All 5 equations share a common point. Manipulating the equations did not change the fact that (2,4) was a solution.
91. You words here...
92. $(3, -2)$
93. $(3,4)$
94. $(-5,10)$
95. ∞ *infinite solutions*
96. $(2, -4)$
97. ∞ *infinite solutions*
98. $(100,200)$
99. $(2220, -1020)$
100. $(16,9)$
101. 17 loonies, 12 toonies
102. 5 is the first number, 3 is the second number.
103. Coffee: $2.05, Cookie: $1.95
104. If sales were $10 000 he would earn the same at both jobs. Straight commission would earn more money when he sold more than $10 000 in merchandise.
105. 430 two-point baskets, 94 three-point baskets
106. There are 20 girls.
107. $4500 at 7%
$7500 at 4%
108. 300 at $2, 270 at $5
109. Oil: $3.50, Gasket: $2.75
110. Frogs: 1.40 x 100grams = 140g
Penguins: 3.60 x 100 grams = 360g
111. Leechi: 3.2 l, Guava: 4.8 l
112. Current: 2 km/h
113. Speeds of swimmer and current are constant.
114. Boat: 27.5 km/h
Current: 22.5 km/h
115. Bumble Bee: 12 km/h
116. Plane: 150 km/h
Wind: 50 km/h
117. 20% stock: 8 l
70% stock: 12 l
118. 20% stock: 2 l
70% stock: 18 l
119. 14K: 12.12 g
18K: 27.88 g
120. 10K: 24.77 g
18K: 25.23 g